Introduction to Landau's Fermi Liquid Theory

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1. Introduction

The principal problem of physics is to determine how bodies behave when they interact. Most basic courses of classical and quantum mechanics treat the problem of one or two particles or bodies. (An external potential can be considered as one very heavy body.) The problem gets more difficult when the number of bodies involved is larger. In particular, in condensed matter we are dealing with a macroscopic number $N \sim 10^{23}$ of particles, and typically hundreds of them directly interact with each other. This problem is commonly known as the *many-body problem*.

There is no general solution to the many-body problem. Instead there is an great number of approximations that successfully explain various limiting cases. Here we discuss one of them, *the Fermi-liquid theory*. This type of approximation for a fermion many-body problem was invented by Landau (1957). It was originally proposed for liquid ^{3}He at very low temperatures. Soon it was realized that a similar approach could be used to other fermion systems, most notably to the conduction electrons of metals. The Fermi-liquid theory allows to understand very many properties of metals. A generalization of the Fermi-liquid theory also allows to understand the superconducting state, which occurs in many metals at low temperatures. Even when Landau's theory is not valid, it forms the standard against which to compare more sophisticated theories. Thus Fermi-liquid theory is a paradigm of many-body theories, and it is presented in detail in many books discussing the many-body problem.

- L.D. Landau and E.M. Lifshitz, *Statistical Physics, Part 2* (Pergamon, Oxford, 1980).
- P. Nozieres, *Theory of interacting Fermi systems* (Bejamin, New York 1964).
- D. Pines and P. Nozieres, *The theory of quantum liquids, Vol. 1 Normal Fermi liquids* (Bejamin, New York 1966).
- G. Baym and C. Pethick, *Landau Fermi-liquid theory* (Wiley, Ney York 1991).

In this lecture I present an introduction to the Landau theory. I present the theory as if it could have logically developed, but this not necessarily reflects historical facts. (If you know historical facts that contradict or support this view, please, let me know.) I will start with the calculation of specific heat in an ideal gas, and compare the result with the measurement in liquid ^{3}He. This leads to the concept of quasiparticles with an effective mass differing from the atomic mass. It is then show that in order to make a consistent theory, one has to allow an interaction between the quasiparticles. Several applications of the Fermi-liquid theory are illustrated. The generalizations of the theory to metals and to the superconducting state are discussed.

2. Preliminary topics

Many-body problem

The many-body problem for indentical particles can be formulated as follows. Consider particles of mass m labeled by index $i = 1, 2, \ldots, N$. Their locations and momenta are written as $\boldsymbol{r}_k$ and $\boldsymbol{p}_k$. The Hamiltonian is

$$H = \sum_{k=1}^{N} \frac{p_k^2}{2m} + V(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots). \tag{1}$$

Here V describes interactions between any particles, and it could in many cases be written as a sum of pairwise interactions $V = V_{12} + V_{13} + \ldots + V_{23} + \ldots$. The classical many-body problem is to solve the Newton's equations.

In quantum mechanics the locations and momenta become operators. In the Schrödinger picture $\boldsymbol{p}_k \to -i\hbar\boldsymbol{\nabla}_k$, where

$$\boldsymbol{\nabla}_k = \hat{\boldsymbol{x}}\frac{\partial}{\partial x_k} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y_k} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z_k}. \tag{2}$$

An additional feature is that generally particles have spin. This is described by an additional index σ that takes values $-s, -s+1, \ldots, s-1, s$ for a particle of spin s. Thus the Hamiltonian operator is

$$H = -\sum_{k=1}^{N} \frac{\hbar^2}{2m}\nabla_k^2 + V(\boldsymbol{r}_1, \sigma_1, \boldsymbol{r}_2, \sigma_2, \ldots). \tag{3}$$

and the state of the system is described by a wave function $\Psi(\boldsymbol{r}_1, \sigma_1, \boldsymbol{r}_2, \sigma_2, \ldots; t)$. Particles having integral spin are called *bosons*. Their wave function has to be symmetric in the exchange of any pairs of arguments. For example

$$\begin{aligned}&\Psi(\boldsymbol{r}_1, \sigma_1, \boldsymbol{r}_2, \sigma_2, \boldsymbol{r}_3, \sigma_3, \boldsymbol{r}_4, \sigma_4, \ldots)\\ &= +\Psi(\boldsymbol{r}_3, \sigma_3, \boldsymbol{r}_2, \sigma_2, \boldsymbol{r}_1, \sigma_1, \boldsymbol{r}_4, \sigma_4, \ldots)\end{aligned} \tag{4}$$

where coordinates 1 and 3 have been exchanged. Particles having half-integral spin are called *fermions*. Their wave function has to be anti-symmetric in the exchange of any pairs of arguments. For example

$$\begin{aligned}&\Psi(\boldsymbol{r}_1, \sigma_1, \boldsymbol{r}_2, \sigma_2, \boldsymbol{r}_3, \sigma_3, \boldsymbol{r}_4, \sigma_4, \ldots)\\ &= -\Psi(\boldsymbol{r}_3, \sigma_3, \boldsymbol{r}_2, \sigma_2, \boldsymbol{r}_1, \sigma_1, \boldsymbol{r}_4, \sigma_4, \ldots)\end{aligned} \tag{5}$$

The quantum many body problem is to solve time-dependent Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t}(\boldsymbol{r}_1, \sigma_1, \ldots, \boldsymbol{r}_N, \sigma_N, t) = H\Psi(\boldsymbol{r}_1, \sigma_1, \ldots, \boldsymbol{r}_N, \sigma_N, t) \tag{6}$$

or to solve energy eigenvalues and eigenstates.

Ideal Fermi gas

As stated in the introduction, there is no general solution of the many-body problem. What one can do is to study some limiting cases. One particularly simple case is *ideal*

gas, where we assume no interactions, $V \equiv 0$. Below we concentrate on ideal spin-half ($s = 1/2$) Fermi gas.

In the absence of interactions we can assume a factorizable form

$$\Psi_0(\boldsymbol{r}_1, \sigma_1, \boldsymbol{r}_2, \sigma_2, \ldots; t) = \phi_a(\boldsymbol{r}_1, \sigma_1)\phi_b(\boldsymbol{r}_2, \sigma_2)\ldots. \tag{7}$$

consisting of a product of single-particle wave functions $\phi_\alpha(\boldsymbol{r}, \sigma)$. This does not yet satisfy the antisymmetry requirement (5) but permuting the arguments in Ψ_0 and summing them all together multiplied by $(-1)^{n_P}$, where n_P is the number of pairwise permutations in a permutation P, one can generate a proper wave function

$$\Psi(\boldsymbol{r}_1, \sigma_1, \boldsymbol{r}_2, \sigma_2, \ldots) = \frac{1}{\sqrt{N}} \sum_P (-1)^{n_P} \times \phi_a(\boldsymbol{r}_{P(1)}, \sigma_{P(1)})\phi_b(\boldsymbol{r}_{P(2)}, \sigma_{P(2)}))\ldots. \tag{8}$$

This is known as Slater determinant since it can also be presented as a determinant

$$\Psi = \frac{1}{\sqrt{N}} \begin{vmatrix} \phi_a(\boldsymbol{r}_1, \sigma_1) & \phi_b(\boldsymbol{r}_1, \sigma_1) & \ldots \\ \phi_a(\boldsymbol{r}_2, \sigma_2) & \phi_b(\boldsymbol{r}_2, \sigma_2) & \ldots \\ \vdots & \vdots & \ddots \end{vmatrix}. \tag{9}$$

We can now see if any two of the single-particle wave functions ϕ_a, ϕ_b, ... are identical, the resulting wave function vanishes identically. (Verify this in the case of two fermions.) This is the *Pauli exclusion principle*, which states that a single state can be occupied by one fermion only.

The natural choice for wave functions of a single free particle are plane wave states. In order to incorporate the spin, we have "spin-up states"

$$\phi_{\boldsymbol{p}\uparrow}(\boldsymbol{r}, \sigma) = \begin{cases} \frac{1}{\sqrt{V}} e^{i\boldsymbol{p}\cdot\boldsymbol{r}/\hbar} & \text{if } \sigma = \frac{1}{2} \\ 0 & \text{if } \sigma = -\frac{1}{2} \end{cases}. \tag{10}$$

and "spin-down states"

$$\phi_{\boldsymbol{p}\downarrow}(\boldsymbol{r}, \sigma) = \begin{cases} 0 & \text{if } \sigma = \frac{1}{2} \\ \frac{1}{\sqrt{V}} e^{i\boldsymbol{p}\cdot\boldsymbol{r}/\hbar} & \text{if } \sigma = -\frac{1}{2} \end{cases}. \tag{11}$$

Here the wave vector $\boldsymbol{k}$ or the momentum $\boldsymbol{p} = \hbar\boldsymbol{k}$ appears as a parameter. In order to count the states, it is most simple to require that the wave functions are periodic in a cube of volume $V = L^3$, which allows the momenta $\boldsymbol{p}$ (n_x, n_y and n_z integers)

$$p_x = \frac{2\pi\hbar n_x}{L}, \quad p_y = \frac{2\pi\hbar n_y}{L}, \quad p_z = \frac{2\pi\hbar n_z}{L}. \tag{12}$$

We suppose that the volume V is very large. Then we can take the limit $V \to \infty$ in quantities that do not essentially depend on V.

The energy of a single-particle states is $\epsilon_p = p^2/2m$. The total energy is this summed over all occupied states

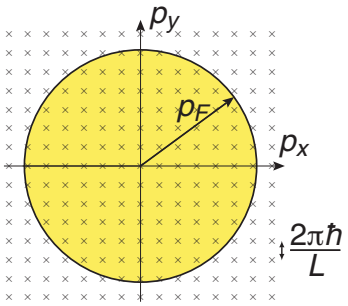
$$E(n_{\boldsymbol{p}\sigma}) = \sum_\sigma \sum_{\boldsymbol{p}} \frac{p^2}{2m} n_{\boldsymbol{p}\sigma} \tag{13}$$

where $n_{\boldsymbol{p}\sigma} = 1$ for an occupied state and is zero othewise.

The ground state of a system with N particles has N lowest energy single-particle states occupied and others empty. Here the maximal kinetic energy of an occupied state is called Fermi energy ϵ_F. We also define the Fermi wave vector k_F and Fermi momentum $p_F = \hbar k_F$ so that

$$\epsilon_F = \frac{p_F^2}{2m} = \frac{\hbar^2 k_F^2}{2m}. \tag{14}$$

In momentum space this defines the Fermi surface ($p = p_F$). All states inside the Fermi surface ($p < p_F$) are occupied, and the ones outside are empty.



The number of particles can be calculated as

$$N = 2 \sum_{p<p_F} 1 = 2\frac{\frac{4}{3}\pi p_F^3}{(2\pi\hbar/L)^3}, \tag{15}$$

where the factor 2 comes from spin. From this we get a relation between the Fermi wave vector and the particle density,

$$\frac{N}{V} = \frac{p_F^3}{3\pi^2\hbar^3}. \tag{16}$$

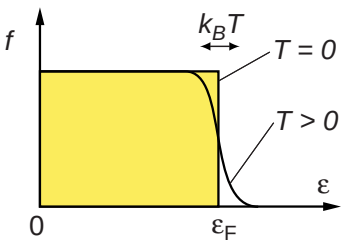
The excited states of the system consist of states where one or more fermions is excited to higher energy states. The average occupations of the states at a given temperature T is given by the Fermi-Dirac distribution

$$f(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)} + 1}. \tag{17}$$

Here $\beta = 1/k_B T$ and $k_\mathrm{B} = 1.38\times10^{-23}$ J/K is Boltzmann's constant, which is needed to express the temperature T in Kelvins, and μ is the chemical potential. At $T = 0$ the system is in its ground state,

$$f(\epsilon) = \begin{cases} 1 & \text{for } \epsilon < \mu \\ 0 & \text{for } \epsilon > \mu. \end{cases} \tag{18}$$

with $\mu = \epsilon_F$. When $T > 0$, the occupation $f(\epsilon)$ gets rounded so that the change from $f \approx 1$ to $f \approx 0$ takes place in the energy interval $\approx k_\mathrm{B}T$.



Next we calculate the specific heat of the ideal Fermi gas at low temperatures. The average energy is given by

$$E = \sum_\sigma \sum_{\boldsymbol{p}} \epsilon_p f(\epsilon_p) = \frac{2}{(2\pi\hbar/L)^3} \int \epsilon_p f(\epsilon_p) d^3p$$
$$= \frac{8\pi}{(2\pi\hbar/L)^3} \int_0^\infty p^2 \epsilon_p f(\epsilon_p) dp \qquad (19)$$

Changing $\epsilon = p^2/2m$ as the integration variable we get

$$\frac{E}{V} = \frac{\sqrt{2m^3}}{\pi^2\hbar^3} \int_0^\infty \epsilon^{3/2} f(\epsilon) d\epsilon$$
$$= \int_0^\infty g(\epsilon)\epsilon f(\epsilon) d\epsilon \qquad (20)$$

In the second line we have expressed the same result by defining a density of states $g(\epsilon) = m\sqrt{2m\epsilon}/\pi^2\hbar^3$.

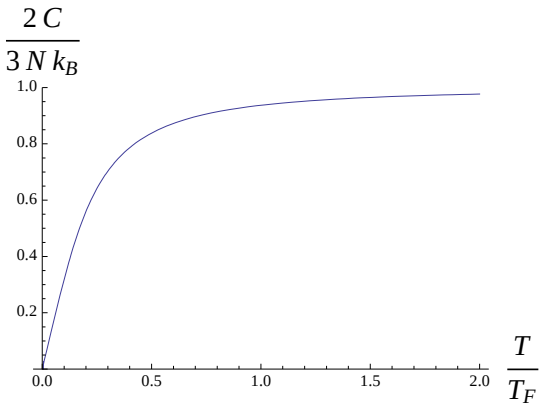
The specific heat is now obtained as the derivative of energy

$$C = \frac{\partial E(T, V, N)}{\partial T} \qquad (21)$$

In order eliminate μ appearing in the distribution function (17) one has to simultaneously satisfy

$$\frac{N}{V} = \int_0^\infty g(\epsilon) f(\epsilon) d\epsilon \qquad (22)$$

The result calculated with Mathematica is shown below



At high temperatures $T \gg T_F$ the Fermi gas behaves like classical gas, where the average energy per particle is $3k_BT/2$, known as the equipartition theorem, and this gives the specific heat $3k_B/2$ per particle. We see that at lower temperatures $T < T_F$ the specific heat is reduced. This can be understood that only the particles with energies close to the Fermi surface can be excited. Those further than energy k_BT from the Fermi surface cannot be excited, and thus do not contribute to specific heat. At temperatures $T \ll T_F$ the specific heat is linear in T:

$$C = \frac{\pi^2}{3} g(\epsilon_F) k_B^2 T + O(T^2). \qquad (23)$$

We see that the linear term is determined by the density of states at the Fermi surface

$$g(\epsilon_F) = \frac{m p_F}{\pi^2\hbar^3}. \qquad (24)$$

(For detailed derivation see Ashcroft-Mermin, Solid state physics.)

Liquid ^{3}He

Helium has two stable isotopes, ^{4}He and ^{3}He. The former is by far more common in naturally occurring helium. It is a boson since in the ground state both the two electrons have total spin zero, and also the nuclear spin is zero. It has a lot of interesting properties that could be discussed, but here we concentrate on the other isotope. ^{3}He is a fermion because the nuclear spin is one half, $s = 1/2$. Studies of ^{3}He were started as it became available in larger quantities in the nuclear age after world war II as the decay product of tritium. The two isotopes of helium are the only substances that remain liquid even at the absolute zero of temperature.

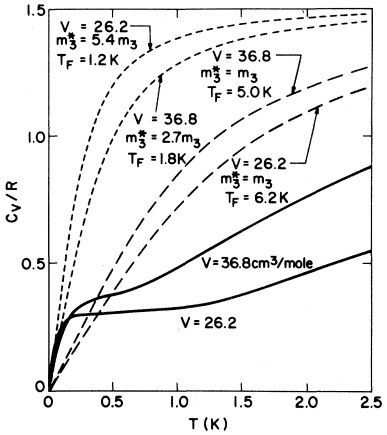


Figure: the specific hear of liquid ^{3}He at two different densities (Greywall 1983).

We see that at low temperatures, the specific heat is linear in temperature. This resembles the ideal gas discussed above, but is quite puzzling since the atoms in a liquid are more like hard balls continuously touching each other! Also, quantitative comparison of the slope with the ideal Fermi gas gives that the measured slope is by factor 2.7 larger.

3. Construction of the theory

Landau's idea

The experiment above raises the following idea. Could it be possible that low temperature liquid ^{3}He would effectively be like an ideal gas? This was the problem Landau started thinking. He had to answer the following questions

- How could dense helium atoms behave like an ideal gas?
- If there is explanation to the first question, how one can understand the difference by 2.7 in the density of states?
- If previous questions have positive answers, are any other modifications needed compared to the ideal gas?

An obvious problem with the ideal gas wave function (8) is that there is too little correlation between the locations $\boldsymbol{r}_k$ of the particles. The Pauli principle prohibits for two particles with the same spin to occupy the same location, but there is no such restriction for particles with opposite spins. Thus it is equally likely to find two opposite-spin particles just at the same place than at any other places in the ideal gas wave function (8).

Weak interactions

As a first attempt to answer the questions, consider point-like particles (instead of real ^{3}He atoms). In an ideal gas the particles fly straight trajectories without ever colliding. If we now allow some small size for the particles, they will collide with each other.

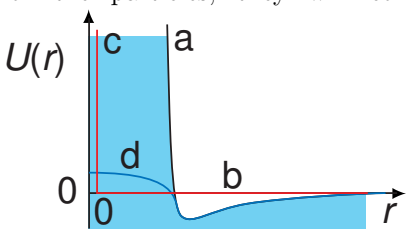


Figure: illustration of various particle-particle potentials $U(r)$: (a) the potential between two ^{3}He atoms, (b) ideal gas potential $U \equiv 0$, (c) potential used in scattering approach, (d) potential used in perturbation theory.

We have to consider two particles with momenta $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ colliding and leaving with momenta $\boldsymbol{p}_1'$ and $\boldsymbol{p}_2'$. In such a process the momentum and energy has to be conserved,

$$\boldsymbol{p}_1 + \boldsymbol{p}_2 = \boldsymbol{p}_1' + \boldsymbol{p}_2' \tag{25}$$

$$\epsilon_1 + \epsilon_2 = \epsilon_1' + \epsilon_2' \tag{26}$$

At least qualitatively the collision rate can be calculated using the golden rule

$$\Gamma = \frac{2\pi}{\hbar} \sum_f |\langle f|H_{\rm int}|i\rangle|^2 \delta(E_f - E_i) \tag{27}$$

We see that the rate is proportional to the number of available final states f. Consider specifically the case of filled Fermi sphere plus one particle at energy $\epsilon_1 > \epsilon_F$. We wish to estimate the allowed final states when particle 1 collides with any particle inside the Fermi sphere, $\epsilon_2 < \epsilon_F$. The final state has to have two particles outside the Fermi sphere ($\epsilon_1' > \epsilon_F$, $\epsilon_2' > \epsilon_F$) since the Pauli principle forbids all states inside. We see that the number final states gets very small when the initial particle is close to the Fermi energy, namely both ϵ_2 and ϵ_1' have to be chosen in an energy shell of thickness $\propto \epsilon_1 - \epsilon_F$. This means that the final states are limited by factor $\propto (\epsilon_1 - \epsilon_F)^2$. Thus the scattering of low energy particles is indeed suppressed and thus resembles the one in ideal gas.

But ^{3}He atoms are not point particles, rather they touch each other continuously. Thus for one particle to move, the others must give the way. This is one of the hardest problems in many body theory even today, but one can get some idea of what happens with a model: instead of true ^{3}He-^{3}He interaction potential, one assumes a weak potential, whose effect can be calculated using quantum-mechanical perturbation theory. We will skip this calculation here (see Landau-Lifshitz). The result is that the excitation spectrum remains qualitatively similar as in free Fermi gas but there is a shift in energies. Consider specifically the case, already mentioned above, of a filled Fermi sphere plus one particle at momentum $\boldsymbol{p}$, with $p > p_F$. This excited state of the ideal gas corresponds to the excitation energy

$$\epsilon_p - \epsilon_F = \frac{p^2}{2m} - \epsilon_F \approx \frac{p_F}{m}(p - p_F) \tag{28}$$

where the approximation is good if p is nof far from the Fermi surface ($p - p_F \ll p_F$). The effect of the weak interactions is now that the excitation energy still is linear in $p - p_F$, but the coefficient is no more p_F/m. It is customary to write the new excitation energy in the form

$$\epsilon_p - \epsilon_F = \frac{p_F}{m^*}(p - p_F) \tag{29}$$

where we have defined the *effective mass* m^*. Note that the Fermi momentum p_F is not changed, equation (16) still remains valid. With the new dispersion relation (29) we get a new density of states

$$g(\epsilon_F) = \frac{m^* p_F}{\pi^2 \hbar^3}. \tag{30}$$

This is determined by the effective mass m^*, not the bare particle mass m as for ideal gas (24). We now see that weak interactions can explain that the specific heat coefficient (23) differs form its ideal gas value. However, the theory is valid for small perturbations, say 10%, and thus is insufficient to explain the factor 2.7.

Quasiparticles

Landau now made the following assumptions. i) Even for strong interactions, the excitation spectrum remains as in (29). Such excitations are called *quasiparticles*: they develop continuously from single-particle excitations when the interactions are "turned on", but they consist of correlated

motion of the whole liquid. ii) The quasiparticles have long life time at low energy, like in the scattering approximation above.

It should be noticed that Landau's theory is phenomenological. At this state it has one parameter, m^*, whose value is unknown theoretically, but can be obtained from experiments.

Although the detailed structure of the quasiparticle remained undetermined, we can develop a qualitative picture with a model. Consider a spherical object moving in otherwise stationary liquid. The details of this model are discussed in the appendix. The main result is that associated with the moving object, there is momentum in the fluid in the same direction. In the literature this is sometimes called "back flow", but I find this name misleading. Rather it should be called "forward flow"or that the moving object drags with itself part of the surrounding fluid. Thinking now that the total momentum of the quasiparticle is fixed, this means that switching on the interactions slows the original fermion down, since part of the momentum goes into the surrounding fluid and less is left for the original fermion.

The same picture is obtained by quantum mechanical analysis. In order to get the velocity of the quasiparticle, we have to form a localized wave packet. This travels with the group velocity. Based on the dispersion relation (29) the group velocity is

$$v_{\rm group} = \frac{dE_p}{dp} = \frac{p_F}{m^*}. \tag{31}$$

This means that the momentum of the original fermion in the interacting system is $mv_{\rm group}\hat{\boldsymbol{p}} = (m/m^*)\boldsymbol{p}$, i.e. the original fermion carries fraction m/m^* of the momentum $\boldsymbol{p}$ and the fraction $1-m/m^*$ is carried by other fermions surrounding the original one.

Quasiparticle interactions

Thus far we have arrived at the picture that the low energy properties of a Fermi liquid can be understood as an ideal gas with the difference that the effective mass m^* appears instead of the particle mass m. In the following we show that this cannot be the whole story, and one more ingredient has to be added in order to arrive at a consistent theory.

A general requirement of any physical theory is that the predictions of the theory should be independent of the coordinate system chosen. In the present case, one has to pay attention to Galilean invariance. That means that the physics should be the same in two coordinate frames that move at constant velocity with respect to each other. To be specific consider a coordinate system O, and a second coordinate system O' that moves with velocity $\boldsymbol{u}$ as seen in the frame O. We assume to study a system of N particles (interacting or not) of mass m. If the total momentum of this system in O' is $\boldsymbol{P}'$, then the momentum seen in frame O has to be $\boldsymbol{P} = \boldsymbol{P}' + Nm\boldsymbol{u}$. Now the Galilean invariance requires that if one determines the state of the system in O' at fixed total momentum $\boldsymbol{P}'$, it is the same as one would do in O with momentum $\boldsymbol{P}$.

The ideal gas obviously has to satisfy Galilean invariance. However, when replace the particle mass m by m^* in (29), the Galilean invariance is broken. The cure for this problem is that we have to allow interactions between the quasiparticles. Thus we rewrite (29) into the form

$$\epsilon_{\boldsymbol{p}} - \epsilon_F = \frac{p_F}{m^*}(p-p_F) + \frac{1}{V}\sum_\sigma \sum_{\boldsymbol{p}} f(\boldsymbol{p},\boldsymbol{p}')(n_{\boldsymbol{p}} - n^{(0)}_{\boldsymbol{p}}). \tag{32}$$

Here $n_{\boldsymbol{p}}$ is the distribution of the quasiparticles, $n^{(0)}_{\boldsymbol{p}} = \Theta(p_F - p)$ is the distribution function in the ground state, where $\Theta(x)$ is the step function ($\Theta(x) = 0$ for $x < 0$ and $\Theta(x) = 1$ for $x > 0$). The function $f(\boldsymbol{p},\boldsymbol{p}')$ describes the interaction energy between two quasiparticles having momenta $\boldsymbol{p}$ and $\boldsymbol{p}'$.

The first thing to notice is that when only one or a few quasiparticles are excited, the interaction term in (32) is negligible since $n_{\boldsymbol{p}} - n^{(0)}_{\boldsymbol{p}} \approx 0$. In this case the excitation energy $\epsilon_{\boldsymbol{p}}$ (32) reduces to its the previous expression (29).

Consider next an uniformly displaced Fermi sphere, $n_{\boldsymbol{p}} = n^{(0)}_{\boldsymbol{p}-m\boldsymbol{u}} = \Theta(p_F - |\boldsymbol{p} - m\boldsymbol{u}|)$. This is the stationary ground state in the O' frame. In order to the theory to be Galilean invariant, the excitation energy in the must be changed from (29) to the ideal gas value (28) at the displaced Fermi surface. As a formula

$$\frac{p_F}{m} m\hat{\boldsymbol{p}}\cdot\boldsymbol{u} = \frac{p_F}{m^*} m\hat{\boldsymbol{p}}\cdot\boldsymbol{u} + \frac{1}{V}\sum_\sigma\sum_{\boldsymbol{p}} f(\boldsymbol{p},\boldsymbol{p}')(n^{(0)}_{\boldsymbol{p}-m\boldsymbol{u}} - n^{(0)}_{\boldsymbol{p}}). \tag{33}$$

We see that this could not be satisfied without the interaction term. In order to work (33) further, we need to study $f(\boldsymbol{p},\boldsymbol{p}')$. Because of spherical symmetry, it can depend only on the relative directions of $\boldsymbol{p}$ and $\boldsymbol{p}'$, i.e. $f(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}', p, p')$. Futher since we are interested only in quasiparticles close to the Fermi surface, we approximate $f(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}', p, p') \approx f(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}', p_F, p_F)$. This means that f depends only on the angle between $\boldsymbol{p}$ and $\boldsymbol{p}'$, i.e. $f(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}')$. In addition, it is conventional to define $F(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}') = g(\epsilon_F) f(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}')$. Such a function can be expanded in Legendre polynomials

$$F(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}') = \sum_{l=0}^{\infty} F^s_l P_l(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}') \tag{34}$$

where $P_0(x) = 1$, $P_1(x) = 1$, $P_2(x) = (3x^2-1)/2$, etc. Using these we can now reduce the requirement (33) to

$$\frac{m^*}{m} = 1 + \frac{F^s_1}{3}. \tag{35}$$

We have arrived at the result that in order to have $m^* \neq m$, we also should include an interaction between the quasiparticles of the form of the F^s_1 term in (34). The other interaction term wiht coefficients F^s_l are not required for internal consistency of the theory but some of them appear as a result of perturbation theory. In order to make general phenomenology, they all should be retained.

Until now we have not considered the fermion spin except that they produced factors of two. In magnetic field the system becomes spin polarized, and this is no more sufficient. All the previous analysis can be generalized to include spin dependence. This means that the quasiparticle energy $\epsilon(\boldsymbol{p})$ becomes a 2×2 spin matrix. Equation (32) has to be generalized to

$$\epsilon_{\eta\nu}(\boldsymbol{p}) - \epsilon_F \delta_{\eta\nu} = \frac{p_F}{m^*}(p - p_F)\delta_{\eta\nu} + \frac{1}{V}\sum_{\boldsymbol{p}} f_{\eta\alpha,\nu\beta}(\boldsymbol{p},\boldsymbol{p}')[n_{\alpha\beta}(\boldsymbol{p}) - n^{(0)}(\boldsymbol{p})\delta_{\alpha\beta}]. \quad (36)$$

where

$$\frac{f_{\eta\alpha,\nu\beta}(\boldsymbol{p},\boldsymbol{p}')}{g(\epsilon_F)} = \sum_{l=0}^{\infty}(F_l^s \delta_{\eta\alpha}\delta_{\nu\beta} + F_l^a \boldsymbol{\sigma}_{\eta\alpha}\cdot\boldsymbol{\sigma}_{\nu\beta})P_l(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}') \quad (37)$$

We see that including the spin dependence the theory has two sets of parameters, F_l^s and F_l^a with $l = 0, 1, \ldots$. (Here s denotes symmetric and a antisymmetric.) One of these parameters, F_1^s is related to m^* by (35).

4. Applications

In normal fermi liquid:

Specific heat (see above)

$$C = \frac{\pi^2}{3} g(\epsilon_F) k_B^2 T + O(T^2). \quad (38)$$

Magnetic susceptibility

$$\chi = \mu_0 \mu_m^2 \frac{g(\epsilon_F)}{1 + F_0^a}, \quad (39)$$

where μ_m is the magnetic moment of a particle.

Sound velocity

$$c = v_F \sqrt{\frac{1}{3}(1 + F_0^s)(1 + \frac{1}{3}F_1^s)}. \quad (40)$$

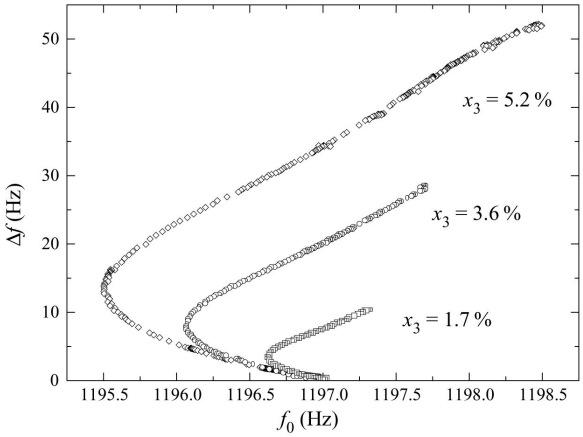
Zero sound

Spin waves

Acoustic impedance

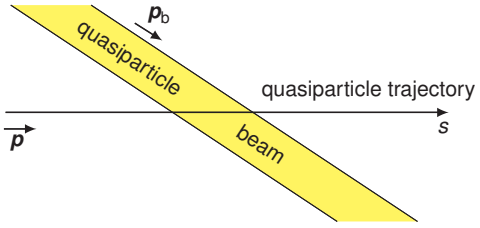
Generalizations to superfluid state: almost all quantities affected by Fermi-liquid interactions.

4.1 Vibrating wire

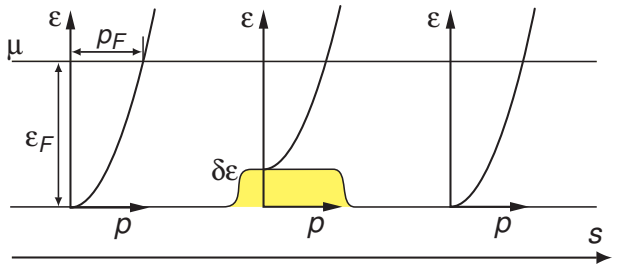


The figure shows the frequency (horizontal) and damping (vertical) of a vibrating parametrically as the mean free path gets bigger with lowering temperature (from small to large damping). The experiment is done in mixtures of ^{3}He and ^{4}He at three different ^{3}He concentrations x_3. Source: Martikainen et al 2002.

One particular problem is to explain why the frequency at in the ballistic limit (lowest temperatures) gets larger than in the high temperature limit. This has to do with quasiparticle interactions.



In order to understand what happened for the propagating quasiparticle, let us consider an arbitrary quasiparticle trajectory crossing a quasiparticle beam. In a Fermi gas, a quasiparticle on the trajectory would not react to the beam at all. In a Fermi liquid it experiences the potential change $\delta\epsilon$ caused by the beam. In the simple case that F_0 is the only relevant Fermi-liquid parameter, the potential $\delta\epsilon = [F_0/2N(0)]n'$ is determined by the particle density in the beam.



The basic assumption of the Fermi-liquid theory is that the potential is small compared to the Fermi energy ϵ_F, i.e. $\delta\epsilon \ll \epsilon_F$. This means that a quasiparticle is slightly decelerated when it enters the beam and it is accelerated back when it leaves the crossing region. In the energy point of view (figure above), the quasiparticle flies at constant energy $\epsilon \approx \mu$ and the potential is effectively compensated by depletion of fermions with the same momentum direction in the crossing region. If there were a second quasiparticle beam along the trajectory, there appears to be no interaction between the two stationary beams.

Let us now consider the case that the intensity of the (original) quasiparticle beam is varying in time. This means that the potential seen on the crossing trajectory changes. This changes the number of particles stored in the crossing region, and thus leads to emission of particle or hole like quasiparticles from the crossing region. This takes place on all crossing trajectories. In the case of a vibrating wire, the motion of the wire generates a beam of quasiparticles. This beam interacts with the quasiparticles that are coming to the wire. In case of negative F_0 this increases the restoring force of the wire and thus also the frequency.

In order to determine this quantitatively, one has to solve the Landau-Boltzmann equation.

$$\frac{\partial n(\boldsymbol{x},\boldsymbol{p},t)}{\partial t} + \frac{\partial n(\boldsymbol{x},\boldsymbol{p},t)}{\partial \boldsymbol{x}} \cdot \frac{\partial \epsilon(\boldsymbol{x},\boldsymbol{p},t)}{\partial \boldsymbol{p}} - \frac{\partial n(\boldsymbol{x},\boldsymbol{p},t)}{\partial \boldsymbol{p}} \cdot \frac{\partial \epsilon(\boldsymbol{x},\boldsymbol{p},t)}{\partial \boldsymbol{x}} = I(n(\boldsymbol{x},\boldsymbol{p},t)) \tag{41}$$

The linearized kinetic equation simplifies to

$$\frac{\partial n_l}{\partial t} + v_{\rm F}\hat{\boldsymbol{p}} \cdot \boldsymbol{\nabla}(n_l + \delta(\epsilon^{(0)}_{\boldsymbol{p}} - \mu)\delta\epsilon) = I, \tag{42}$$

$$\frac{\partial \phi}{\partial t} + v_{\rm F}\hat{\boldsymbol{p}} \cdot \boldsymbol{\nabla}(\phi + \delta\epsilon) = I. \tag{43}$$

Appendix: hydrodynamic model of a quasiparticle

The starting point is Euler's equation and the equation of continuity

$$\begin{aligned} \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} &= -\frac{1}{\rho}\boldsymbol{\nabla}p \\ \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) &= 0, \end{aligned} \tag{44}$$

where p is now the pressure. We assume a small velocity so that the nonlinear term can be dropped and assume incompressible fluid, ρ = constant. The boundary condition on the surface of a sphere of radius a is $\boldsymbol{n}\cdot\boldsymbol{v} = \boldsymbol{n}\cdot\boldsymbol{u}$, where $\boldsymbol{u}$ is the velocity of the sphere and $\boldsymbol{n}$ the surface normal. We assume that far from the sphere $\boldsymbol{v} \to 0$. The velocity can be represented using the potential $\boldsymbol{v} = \nabla\chi$ where $\chi = -a^3\boldsymbol{u}\cdot\boldsymbol{r}/2r^3$. By the Euler equation the pressure $p = -\rho\dot{\chi}$, and the force exerted by the sphere on the fluid

$$\boldsymbol{F} = \int d\boldsymbol{a}\, p = \frac{2\pi a^3\rho}{3}\dot{\boldsymbol{u}} \tag{45}$$

is proportional to the acceleration $\dot{\boldsymbol{u}}$. Therefore, associated with a moving sphere, there is momentum in the fluid

$$\boldsymbol{p} = \frac{2\pi a^3\rho}{3}\boldsymbol{u} \tag{46}$$

corresponding to half of the fluid displaced by the sphere and moving in the same direction and at the same velocity as the sphere. Note that this may differ from the total momentum of the fluid, which is not essential here, and is undetermined in the present limit of unlimited fluid.